THE VERTICAL CYLINDER WITH HORIZONTAL TOP SURFACE

Equation (25) describes also the H-averaged heat transfer coefficient on the vertical surface of the cylinder shown on the right-hand side of Fig. 2. What changes in the present case is only the expression for parameter B, which follows from the condition of mass continuity over the circular edge of the top surface

$$\frac{1}{\pi D}\dot{m}\left(\frac{D}{2}\right) = \Delta(0). \tag{28}$$

After using equations (19), (20), (23) and (24), we obtain

$$B = 0.259\left(\frac{D}{H}\right)^{4/5}\Pi_H^{-1/15}. \tag{29}$$

This B expression is similar to the one for the vertical face of a slab with a flat top, equation (27). Again, the effect of the flow rate of condensate produced by the disc-shaped top surface is to decrease the condensation rate that would have been produced by the vertical surface in the absence of the disc. This effect is presented graphically in Fig. 2, the abscissa of which reveals a new dimensionless group that can be written alternatively as $(D/H)\Pi_H^{-1/12}$

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Mixed forced and natural convection from two-dimensional or axisymmetric bodies of arbitrary contour

M. R. CAMERON,† D. R. JENG‡ and K. J. DEWITT§

† Polymer Institute, The University of Toledo, Toledo, OH 43606, U.S.A.
‡ Department of Mechanical Engineering, The University of Toledo, Toledo, OH 43606, U.S.A.
§ Department of Chemical Engineering, The University of Toledo, Toledo, OH 43606, U.S.A.

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1. INTRODUCTION

MIXED convection problems involving laminar boundary layers have been treated in a variety of ways. Most solutions found thus far have been valid only over a limited range of the buoyancy parameters, i.e. they represent situations perturbed from either the pure forced or natural convection cases. Two sets of solutions must be obtained in order to have valid results for the entire range. Some of the early studies on mixed convection have dealt only with similarity solutions. For example, Sparrow *et al.* [1] pointed out that for an isothermal wedge, a similarity solution only existed for a wedge angle of 120°. Mixed convection for the vertical and the horizontal plate have also been discussed by Schneider [2], Lloyd and Sparrow [3], Sparrow and Minkowycz [4], Chen *et al.* [5], Ramachandran *et al.* [6] and Raju *et al.* [7]. For the horizontal plate, the momentum equation in the direction normal to the plate must be accounted for in order to obtain meaningful results. The integral of the temperature function adds complexity to the numerical solution when this momentum equation is included. Solution techniques have been mostly local similarity or local non-similarity in nature. Wedge flow was analyzed by Gunness and Gebhart [8]. They perturbed the Falkner–Skan equation for a situation including the buoyancy effects in directions both along and normal to the surface. Their perturbation quantities, however, were related to the buoyancy parameter which limited the results to relatively small effects. Mixed convective flow for inclined plate and sphere geometries were investigated by Mucoglu and Chen [9, 10]. Both local non-similarity [11, 12] and Keller and Cebeci's [13] finite-difference algorithm were used to solve the laminar boundary layer equations which represented the system perturbed from forced and/or free convection. A good reference which summarizes the literature for mixed convection is the new text of Gebhart *et al.* [14].

In the present study, an analysis is made for mixed convection in laminar boundary layer flow over two-dimensional or axisymmetric isothermal surfaces with arbitrary contour. The laminar boundary layer equations for mixed convection are transformed and formulated in such a way that they are valid over the entire range of concern, from pure forced convection to pure natural convection. The Merk–Chao series [15, 16] for two parameters is developed in this paper and is applied to obtain the solution for mixed convection. By introducing this two-parameter Merk–Chao series into the transformed boundary layer equations, there results a set of ordinary differential equations with two parameters which implicitly absorb the geometry and orientation of the surface. Therefore, by assigning numerical values to these parameters, this set of equations can be solved so that the results for the flow field and the heat transfer can be expressed in terms of universal functions.

The purpose of this study was (i) to obtain a set of transformations which would allow the boundary layer equations to be solved for the entire mixed convection range, (ii) to adapt the Merk–Chao expansion to these mixed convection equations, and (iii) to formulate modified definitions for the friction factor group and the Nusselt number group which are finite for the full mixed convection range. Section 2 of this paper illustrates the development of the transforms plus it presents the expansion used. A comparison of the solutions to the resultant equations with previous investigations is covered in Section 3. Finally, the modified definitions for

NOMENCLATURE

C_f	local friction factor
$f(\xi, \eta)$	dimensionless stream function
Gr_x	Grashof number
g_x	x-component of gravitational force
k	thermal conductivity
h	convective heat transfer coefficient
Nu_z	local Nusselt number, hz/k
Pr	Prandtl number
r	for axisymmetric flow, this is the normal distance from the axis of symmetry to the body surface
R	radius of sphere or cylinder
R_x	convection ratio defined for infinite geometries
R_R	convection ratio defined for finite geometries
Re_z	Reynolds number generalized for mixed flow, $U_R z/\nu$
T	fluid temperature in the boundary layer
T_∞	temperature in the free stream
T_w	temperature at the surface
u, v	velocity components in x- and y-directions, respectively
u_∞	velocity in the free stream
u_e	velocity at the outer edge of the boundary layer, due to pressure forces
u_a	pseudo-velocity due to body forces
U	mixed pseudo-velocity function, a function of both forced and natural convection
U_R	reference velocity used in defining the Reynolds number and friction factor
x	coordinate measured along surface
y	coordinate measured normal to surface
z	reference length.

Greek symbols

α	thermal diffusivity
β	coefficient of thermal expansion
γ	angle of inclination for the inclined plate to the vertical
η	pseudo-similarity variable
θ	dimensionless temperature, $(T-T_\infty)/(T_w-T_\infty)$
Λ_B, Λ_3	mixed convection boundary layer parameters, analogous to the wedge parameter for forced convection boundary layers
μ	shear viscosity
ν	kinematic viscosity, μ/ρ
ξ	dimensionless distance along the surface
ρ	fluid density
τ_w	shear stress at surface
ϕ	angular position on sphere or cylinder
Ψ	stream function.

the dimensionless groups are presented and discussed in Section 4.

2. MATHEMATICAL FORMULATION

Consideration is given to the steady, laminar, mixed convective boundary layer flow over two-dimensional or axisymmetric bodies of uniform surface temperature, T_w, situated in a flow field with a free stream temperature of T_∞. The coordinate x is the distance measured along the surface from the lower stagnation point, and y is the distance along the outer normal to the body. The corresponding velocity components are u and v. For rotationally symmetric bodies, $r(x)$ is the radial distance measured from the axis of symmetry to the surface of the body. Constant fluid properties are assumed, except for the density in the buoyancy term. Furthermore, the viscous dissipation term and the buoyancy force in the y-direction are neglected. It is recognized that the current results will not be applicable to the horizontal plate problem or for geometries where the y-buoyancy force is comparable in magnitude to the x-buoyancy force. Note however that the purpose of this study is to illustrate the development of a double series expansion for obtaining the solution to mixed convection problems. The application of this technique is identical whether or not y-buoyancy terms are accounted for. A more in-depth discussion concerned with including the y-buoyancy terms can be found in ref. [17].

The governing boundary layer equations for mixed flow are:

continuity

$$\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial y}=0; \tag{1}$$

momentum

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g_x\beta(T-T_\infty)+u_e\frac{du_e}{dx}+\nu\frac{\partial^2 u}{\partial y^2}; \tag{2}$$

energy

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}; \tag{3}$$

with the boundary conditions

$$u=v=0;\quad T=T_w \quad \text{at} \quad y=0 \tag{4}$$

$$u\to u_e(x);\quad T\to T_\infty \quad \text{at} \quad y\to\infty \tag{5}$$

$$u=u_e(0);\quad T=T_\infty \quad \text{at} \quad x=0 \text{ for all } y. \tag{6}$$

In the above, $r = 1$ for two-dimensional flow. The velocity $u_e(x)$ is the flow velocity along the outer edge of the boundary layer which is assumed to be known either from inviscid flow theory or from experimental measurement. The continuity equation is identically satisfied by introducing a stream-function. The entrance condition (6) is valid only for aiding flow and only aiding flow will be investigated in this paper.

A pseudo-velocity function $U(x)$ is defined by the equation

$$U\frac{dU}{dx}=u_e\frac{du_e}{dx}+g_x\beta(T_w-T_\infty)=u_e\frac{du_e}{dx}+u_a\frac{du_a}{dx}. \tag{7}$$

This form is proposed from the physical consideration that the sum of the driving forces per unit mass due to the streamwise pressure gradient and buoyancy are equated to a pseudo-inertia force. By integrating equation (7), there results

$$U=\sqrt{(u_e^2+u_a^2)}. \tag{8}$$

The replacement of the buoyancy force by a pseudo-velocity function, u_a, was first proposed by Lin and Chao [18] in their analysis of pure natural convection. The pseudo-velocity U strictly depends upon the geometry and orientation of the surface. An alternative form of U which has also been considered is a linear combination of u_e and u_a. Solutions using other forms for the pseudo-velocity were obtained [17], but only the case using equation (8) will be reported in this paper.

To solve the momentum and energy equations, the following coordinate transformations are introduced:

$$x \Rightarrow \xi = \nu \int_0^x r^2 U \, dx = \nu \int_0^x r^2 \sqrt{(u_e^2 + u_a^2)} \, dx \quad (9)$$

$$y \Rightarrow \eta = \frac{Ur}{(2\xi)^{1/2}} y \quad (10)$$

$$\psi = (2\xi)^{1/2} f(\xi, \eta); \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}. \quad (11)$$

These transformations, although not identical to those used by Raju *et al.* [7], perform the same function by allowing the resultant equations to be solved for the full mixed convection range. In the following, the primes denote differentiation with respect to η. The transformed equations become:

momentum equation

$$f''' + ff'' + \Lambda_3 \{\Lambda_B^2 - (f')^2 + (1 - \Lambda_B^2)\theta\} = 2\xi \frac{\partial(f, f')}{\partial(\eta, \xi)} + 2\xi \Lambda_B \frac{d\Lambda_B}{d\xi}(\theta - 1) \quad (12)$$

with the boundary conditions

$$f(\xi, \eta = 0) = f'(\xi, \eta = 0) = 0; \quad f'(\xi, \eta \to \infty) = \Lambda_B \quad (13)$$

where

$$\Lambda_B = \frac{u_e}{U}, \quad \Lambda_3 = 2\xi \frac{1}{u_e^2 + u_a^2}\left[u_e \frac{du_e}{d\xi} + u_a \frac{du_a}{d\xi}\right] = 2\xi \frac{1}{U}\frac{dU}{d\xi}; \quad (14)$$

energy equation

$$\theta'' + Pr\, f\theta' = Pr\, 2\xi \frac{\partial(f, \theta)}{\partial(\eta, \xi)} \quad (15)$$

with the boundary conditions

$$\theta(\xi, \eta = 0) = 1; \quad \theta(\xi, \eta \to \infty) = 0. \quad (16)$$

For the equation sets representing either pure forced flow or pure natural convection, there is only one parameter in the momentum equation. The Merk–Chao series can then be expressed in terms of a single perturbation quantity. However, for mixed convection there are two independent parameters, Λ_3 and Λ_B. Consequently, the coordinate ξ must be expressed as a function of both parameters in the model, and a dual perturbation series results. $f(\xi, \eta)$ is expanded in the following series form as:

$$f = f_0 + 2\xi \frac{d\Lambda_3}{d\xi} f_1 + 2\xi \frac{d\Lambda_B}{d\xi} f_2 + (2\xi)^2 \frac{d^2\Lambda_3}{d\xi^2} f_3 + (2\xi)^2 \frac{d^2\Lambda_B}{d\xi^2} f_4 + (2\xi)^2 \left(\frac{d\Lambda_3}{d\xi}\right)^2 f_5 + (2\xi)^2 \left(\frac{d\Lambda_B}{d\xi}\right)^2 f_6 + (2\xi)^2 \left(\frac{d\Lambda_3}{d\xi}\right)\left(\frac{d\Lambda_B}{d\xi}\right) f_7 + \cdots \quad (17)$$

with the variable $f = f(\xi, \eta)$ and $f_i = f_i(\Lambda_3, \Lambda_B, \eta)$.

A similar expansion is used for the dimensionless temperature function, θ. These expansions are substituted into equations (12) and (15) and terms containing similar perturbation quantities are collected. Only the first set of equations is shown here, but five sets were used in obtaining the solutions

$$f_0''' + f_0 f_0'' + \Lambda_3 \{(\Lambda_B^2 - (f_0')^2) + (1 - \Lambda_B^2)\theta_0\} = 0 \quad (18)$$

$$\theta_0'' + Pr\, f_0 \theta_0' = 0 \quad (19)$$

with the boundary conditions

$$f_0(\xi, 0) = f_0'(\xi, 0) = 0; \quad f_0'(\xi, \infty) = \Lambda_B$$
$$\theta_0(\xi, 0) = 1; \quad \theta_0(\xi, \infty) = 0. \quad (20)$$

The set of simultaneous differential equations for the f_i's and θ_i's were integrated using a fourth-order Runge–Kutta procedure on a VAX 11/780 computer. Details of the procedure can be found in ref. [17]. Computations were made for Λ_3 ranging from 0.0 to 1.5 and Λ_B ranging from 0.0 to 1.0 for Prandtl numbers of 0.7 and 7.0. These solutions are for universal functions and can be applied to calculating the flow and heat transfer over any arbitrary geometry. The case of $\Lambda_B = 1$ corresponds to pure forced convection and, as such, can be compared to the results of Chao and Fagbenle [16]. For the first-order equations, the results match for the first four significant figures. Higher order functions (f_i, θ_i) occasionally differ by as much as 10%, but over most of the range for Λ_3 the difference is 1% or less. For a series which converges rapidly, the higher order terms are much smaller than the first-order term and have little effect on the overall solution. For pure natural convection ($\Lambda_B = 0$), the degree of match for the first-order functions (f_0, θ_0) is within 1% for $Pr = 0.7$ when compared to Lin and Chao [18]. Higher order solutions could not be compared individually because of a slight difference in the forms of the Merk–Chao expansions used.

Once the numerical values for $f_i''(\xi, \eta = 0)$ and $\theta_i'(\xi, \eta = 0)$ are available, the calculations of the local surface shear stress and heat transfer become a simple matter. With U_R as the reference velocity and z the reference length, the Reynolds number, the local friction factor and the local Nusselt number are defined by

$$Re_z = \frac{U_R z}{\nu}; \quad C_f = \frac{2\tau_w}{\rho U_R^2}; \quad Nu_z = \frac{hz}{k}. \quad (21)$$

Thus

$$C_f\, Re_z^{1/2} = \left[\frac{2}{\bar{Q}_s}\right]^{1/2} \left[\frac{U}{U_R}\right]^{3/2} \left[\frac{z}{x}\right]^{1/2} f''(\xi, \eta = 0) \quad (22)$$

and

$$Nu_z\, Re_z^{-1/2} = -\left[\frac{1}{2\bar{Q}_s}\right]^{1/2} \left[\frac{U}{U_R}\right]^{1/2} \left[\frac{z}{x}\right]^{1/2} \theta'(\xi, \eta = 0) \quad (23)$$

where

$$Q_s = \frac{\int_0^x r^2 U \, dx}{r^2 U}; \quad \bar{Q}_s = \frac{1}{x} Q_s. \quad (24)$$

3. CALCULATIONS FOR SPECIFIC GEOMETRIES

This section presents the results of the solution to the equations as applied to several geometries. It will be shown that the solutions to the modified Merk–Chao series do indeed converge fairly rapidly and accurately. In order to compare these results with those in the literature, it is necessary to use the definitions defined in the specific literature for the friction factor group and the Nusselt number group. The situation analyzed first is the inclined plate. The simple forms for the forced convection velocity, $u_e(x) = u_\infty$, and the free convection pseudo-velocity, $u_a(x) = [2g_x\beta(T_w - T_\infty)x]^{1/2}$, make it relatively easy to compute the analytical representations of the parameters and their derivatives needed for the modified Merk–Chao series expansion. They are

$$\sigma = \frac{u_a}{u_e} = \sqrt{\left(2\frac{Gr_x}{Re_x^2}\right)};$$

thus

$$\Lambda_B = \frac{1}{(1 + \sigma^2)^{1/2}} \quad \text{and} \quad 0 \leqslant \Lambda_B \leqslant 1$$

$$2\xi \frac{d\Lambda_B}{d\xi} = -\left(\frac{2}{3}\right)\Lambda_B(1 - \Lambda_B^3);$$

$$(2\xi)^2\frac{d^2\Lambda_B}{d\xi^2} = \left(\frac{2}{3}\right)^2 4\Lambda_B(1-\Lambda_B^3)^2;$$

$$\Lambda_3 = \left(\frac{2}{3}\right)(1-\Lambda_B^3); \quad 2\xi\frac{d\Lambda_3}{d\xi} = \left(\frac{2}{3}\right)^2 3\Lambda_B^3(1-\Lambda_B^3);$$

$$(2\xi)^2\frac{d^2\Lambda_3}{d\xi^2} = -\left(\frac{2}{3}\right)^3 18\Lambda_B^3(1-\Lambda_B^3)^2;$$

$$\bar{Q}_s = \frac{3(1+\Lambda_B)}{2(1+\Lambda_B+\Lambda_B^2)}. \tag{25}$$

The x-component of gravitation is $g_x = g\cos(\gamma)$, where γ represents the angle between the plate and the vertical. Conventional definitions for the friction factor group and the Nusselt number group were used with the reference length z being distance, x, along the surface from the stagnation point and the reference velocity being the free stream velocity, u_∞.

Table 1 shows both the convergence of the Merk–Chao series and also compares our results with those of Mucoglu and Chen [9]. It is felt that by terminating the series at five terms, adequate convergence is demonstrated. The solutions were combined as if consisting of three groups—the local similarity solution, a first-order correction (the sum of the terms multiplied by $2\xi\ d\Lambda_B/d\xi$ and $2\xi\ d\Lambda_3/d\xi$), and a second-order correction (the sum of the two higher order terms). The percentage difference columns in Tables 1 and 2 are based on the solution with five terms in the modified Merk–Chao expansion. Except for small values of the buoyancy parameter, the results differ by less than 1%. For the local Nusselt number, Table 2, the five-term result seems to be off by a fixed percentage over most of the range. Both the solutions to the (f_i, θ_i) equations and the coefficients in the Merk–Chao series appear to decrease for the higher order terms. The series converges very rapidly.

For the horizontal cylinder the forced convection velocity and free convection pseudo-velocity are expressed as

$$u_e = 2u_\infty \sin\phi \quad \text{where} \quad \phi = x/R$$

$$u_a = [2g\beta(T_w - T_\infty)R(1-\cos\phi)]^{1/2}. \tag{26}$$

The Nusselt number group reported in Sparrow and Lee [19] should be multiplied by the square root of 2 in order to compare with our definition. Numerical results matched very well for the entire range of the buoyancy parameter for $\phi = 0°$. For flow around a sphere, the form for the natural convection pseudo-velocity is identical to that for the horizontal cylinder and the forced convection velocity expression, u_e, differs only in that the constant factor is 1.5 vs 2.0. The variable r is given by

$$r(x) = R\sin(x/R) = R\sin\phi \tag{27}$$

where R is the radius of the sphere. As with the horizontal cylinder computations, numerical methods were used to get values for the parameters and the derivatives needed for the modified Merk–Chao expansion. The convergence again is very good. The solutions based on one, three and five terms of the series are indistinguishable up to angular positions of 70° or more. The results of Chen and Mucoglu [10] are presented graphically in ref. [17] and all data read from their figures fell on our curves.

4. PROPOSED DIMENSIONLESS GROUPS FOR MIXED CONVECTION

When using conventional definitions, it is apparent that the local friction factor and Nusselt number get infinitely large as the buoyancy factor, Gr_x/Re_x^2, approaches infinity. These definitions of the local friction factor group and Nusselt number group are more useful for cases slightly perturbed from forced convection. If the entire range of mixed convection is to be examined, more comprehensive definitions are desired for these dimensionless groups which are finite for the whole range of mixed convection. These newly defined dimensionless groups should also reduce in the limits to those currently used for pure forced and natural convection.

Infinite geometries such as the inclined plate are handled slightly different from finite geometries such as the sphere. For infinite bodies, the reference velocity, U_R, recommended is equivalent to the pseudo-velocity U ($= [u_e^2+u_a^2]^{1/2}$). Position along the surface of the body is taken as the reference length. It is instructive to compare numerically the modified

Table 1. Comparison using the conventional friction factor group showing convergence of the Merk–Chao series for an inclined plate ($U^2 = u_e^2+u_a^2$, $Pr = 0.7$, uniform surface temperature)

	Mucoglu and Chen [9]		Current study—$C_f\,Re_x^{1/2}$			Percentage difference
Gr_x/Re_x^2	$f''(0)$	$C_f\,Re_x^{1/2}$	one term	three terms	five terms	(%)
0.0	0.332	0.664	0.664	0.664	0.664	0.0
1.0	1.203	2.406	2.549	2.474	2.462	−2.33
2.0	1.860	3.720	3.827	3.771	3.765	−1.24
3.0	2.439	4.878	4.951	4.908	4.908	−0.61
4.0	2.971	5.942	5.988	5.953	5.956	−0.23
5.0	3.470	6.940	6.963	6.935	6.941	−0.01

Table 2. Comparison of the conventional Nusselt number group for an inclined plate between the current study and the literature ($U^2 = u_e^2+u_a^2$, uniform surface temperature)

	$Pr = 0.7$			$Pr = 7.0$		
Gr_x/Re_x^2	Mucoglu and Chen [9]	Current study	Percentage difference (%)	Mucoglu and Chen [9]	Current study	Percentage difference (%)
0.0	0.2927	0.2927	0.00	0.6459	0.6459	0.00
1.0	0.4088	0.4102	−0.34	0.8612	0.8724	−1.30
2.0	0.4641	0.4664	−0.50	0.9740	0.9852	−1.15
3.0	0.5035	0.5064	−0.58	1.055	1.067	−1.13
4.0	0.5348	0.5382	−0.64	1.121	1.133	−1.07
5.0	0.5661	0.5649	−0.68	1.176	1.189	−1.10

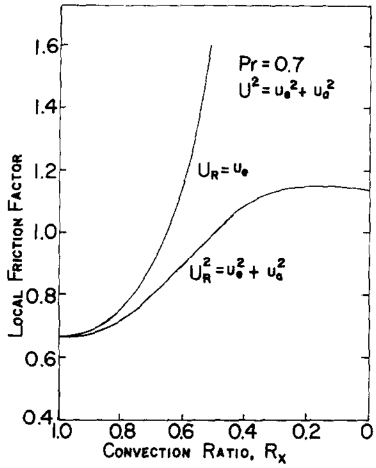

FIG. 1. Effect of reference velocity definition on the local friction factor for an inclined plate.

definition ($U_R \equiv U$) with the conventional definition (equivalent to letting $U_R = u_e$) for forced convection. This is done here using the geometry for an inclined plate. The expressions for the friction factor group and Nusselt number group become

$$C_f\, Re_x^{1/2} = \left[\frac{6}{2+R_x}\right]^{1/2}\left[\frac{U}{U_R}\right]^{3/2} f''(\xi,\eta=0) \quad (28)$$

$$Nu_x\, Re_x^{-1/2} = -\left[\frac{3}{2(2+R_x)}\right]^{1/2}\left[\frac{U}{U_R}\right]^{1/2} \theta'(\xi,\eta=0). \quad (29)$$

With this change in formulation, the data is presented as functions of the convection ratio R_x in Fig. 1. The convection ratio is defined by

$$R_x = \frac{u_e}{u_e+u_a} \text{ which for an inclined plate} = \left[1+\frac{Gr_x}{Re_x^2}\right]^{-1}. \quad (30)$$

This ratio is similar to the mixed flow parameter used by Raju *et al.* [7]. The uppermost curve is based on the forced convection velocity as the reference velocity. The other line represents the modified friction factor group. As expected, the modified definition of the friction factor becomes equal to $(2)^{1/2}f''(\xi, 0)$ for pure forced convection, and equal to $(3)^{1/2}f''(\xi, 0)$ for pure natural convection. This last value is

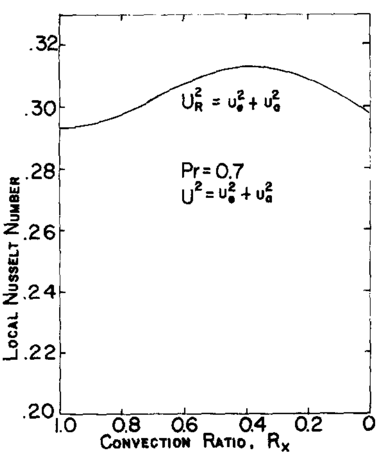

FIG. 2. Effect of reference velocity definition on the local Nusselt number for an inclined plate.

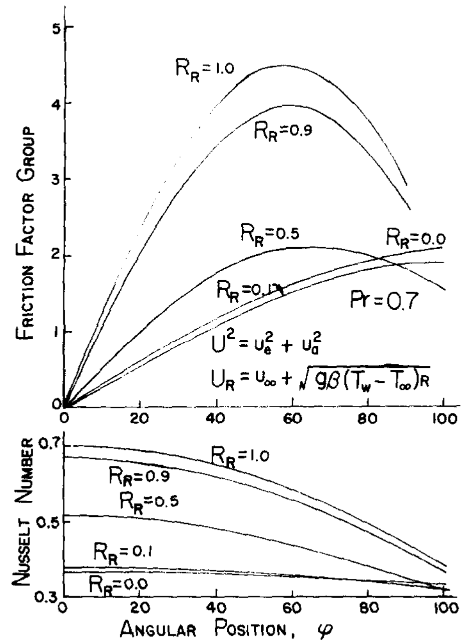

FIG. 3. Friction factor group and Nusselt number group vs angular position—horizontal cylinder.

identical to the limiting case for the situation perturbed from natural convection. Figure 2 illustrates the local Nusselt number group based on the new definition. These dimensionless groups are correlated by the following two equations in terms of the convection ratio R_x for $Pr = 0.7$:

$$C_f\, Re_x^{1/2} = 1.1416 - 0.4775R_x - 0.3013R_x(R_x-1) - 2.414R_x(R_x^2-1) + 2.017R_x(R_x^3-1) \quad (31)$$

and

$$Nu_x\, Re_x^{-1/2} = 0.2970 - 0.0043R_x - 0.4359R_x(R_x-1) - 0.1578R_x(R_x^2-1) + 1.251R_x(R_x^3-1). \quad (32)$$

For geometries involving finite bodies such as the horizontal cylinder or the sphere, where there is a convenient reference length such as the radius R, the convection ratio is defined such that it is *not* a function of position

$$R_R = \frac{u_\infty}{u_\infty + \sqrt{(g\beta(T_w - T_\infty)R)}}. \quad (33)$$

To avoid the local dimensionless groups from being infinite at the stagnation point, the reference velocity U_R is chosen to be

$$U_R = u_\infty + [g\beta(T_w - T_\infty)R]^{1/2}. \quad (34)$$

For both the horizontal cylinder and sphere, the parameters and their derivatives were computed numerically. Plots of these quantities can be found elsewhere. Figure 3 shows the redefined friction factor and Nusselt number for the horizontal cylinder for $Pr = 0.7$ as functions of angular position, ϕ, with the convection ratio, R_R, as a fixed parameter. Table 3 lists some results on the convergence of the series. Shown are the ratios of the similarity solution (one term) and the sum of three terms to the five-term solution for various combinations of convection ratio and angular position ϕ. The closer the ratio is to 1.0, the better the convergence. For increasing angle and increasing forced convection contribution, the convergence worsens but is still very good except for large angles ($\phi \geq 90°$) under conditions of forced flow dominating ($\Lambda_B \approx 1$). Figure 4 depicts the redefined friction factor and Nusselt number for the sphere.

Table 3. Relative convergence of the modified Merk–Chao series expansion as applied to a horizontal cylinder ($Pr = 0.7$, $U^2 = u_e^2 + u_a^2$)

ϕ (deg)	R_R	C_{f1}/C_{f5}	C_{f3}/C_{f5}	Nu_1/Nu_5	Nu_3/Nu_5
0	1.0	1	1	1	1
45	1.0	0.9884	1.0005	1.006	0.999
75	1.0	0.9207	1.0245	1.013	0.979
90	1.0	0.646	1.363	0.9486	0.8950
90	0.9	0.7250	1.271	0.9651	0.9029
90	0.5	0.9601	1.0073	1.0326	0.9845
90	0.1	0.9817	1.0011	1.0030	0.9991

5. CONCLUSIONS

The technique for applying a Merk–Chao series expansion to boundary layer problems has been extended to mixed convection flow over two-dimensional and axisymmetric bodies. This technique involves a double perturbation expansion about a local similarity state. The resulting ordinary differential equations are then amenable to solution via a shooting method. The resulting universal functions are independent of geometry and can be tabulated with respect to the two mixed convection parameters. To insure that the solutions to the boundary layer equations would be valid for the whole range of the mixed flow region, appropriate coordinate transformations, which are functions of both forced and natural convection velocities, were first applied to the energy and momentum equations. The method was adapted to, but is not limited to, steady, laminar, incompressible flow over isothermal surfaces.

In order to present the data over the entire convection range, a convection ratio, which was chosen to be the ratio of forced to the sum of forced plus natural convection reference velocities, was introduced. In addition, the definitions for the Reynolds number and friction factor were redefined based upon choosing a suitable reference velocity. Several geometries were considered. For an inclined plate, the series can be shown to converge very rapidly. For both the sphere and horizontal cylinder, convergence was very good except for angular positions of 90° or more and forced flow dominating. Under this condition, more terms in the series may be required to obtain more accurate numerical results.

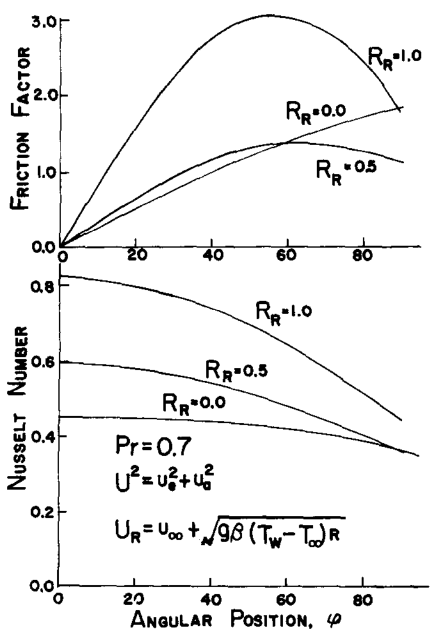

FIG. 4. Friction factor group and Nusselt number group vs angular position—sphere.

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